Appendix B. Kirchhoff and Stolt migration

B.1. Kirchhoff migration

The Kirchhoff migration technique finds its origin in the field of seismics. Although the Kirchhoff migration has been developed for the backpropagation of scalar pressure wavefields, it is often applied (with success) to electromagnetic waves. The basic idea in the Kirchhoff-migration is to back-propagate the scalar wavefront, measured in the data-plane (as defined in the exploding source model), to the object plane at $t = 0$, using an integral solution method to the scalar wave equation.

Suppose a scalar field $b(\bar{r},t)$, satisfying the scalar 3D wave equation

$$\nabla^2 b(\bar{r},t) - me\frac{\partial^2}{\partial t^2} b(\bar{r},t) = 0 \qquad \textbf{(B. 1)}$$

The configuration and the co-ordinate system are shown in Fig. 1. The array of receivers recording the data $b(\bar{r}',t')$ in the data plane at the air-ground interface (denoted here as S'), will be replaced by an array of secondary sources, each driven in reverse time by the recorded data. In the configuration:

$\bar{r}$ is the co-ordinate of the observer,

t the real time of the observer,

$\bar{r}'$ the co-ordinate of the sources,

t' the real time of the sources,

S' the data plane, containing the secondary sources

$b(\bar{r},t)$ the scalar wavefield

$b(\bar{r}',t')$ represents the measurements of this scalar field in the data plane.

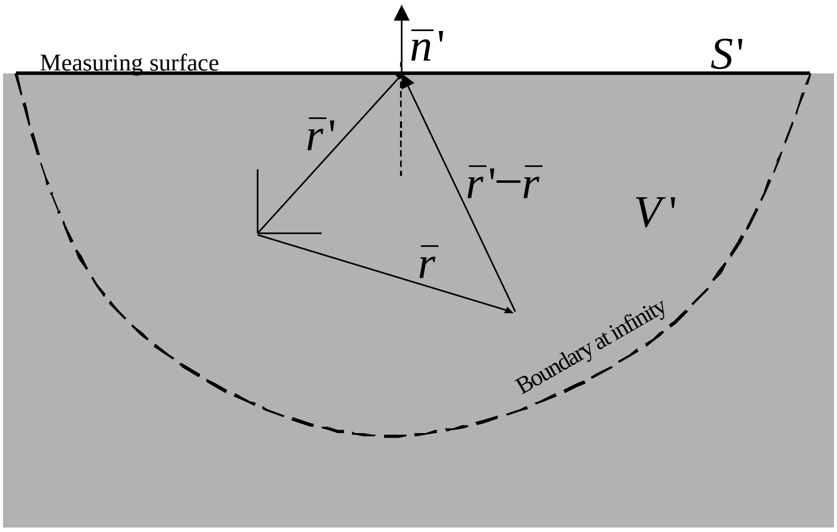

Fig. 1: Configuration and co-ordinate system for the Kirchhoff migration

In seismics, the air-ground interface is considered as a perfect reflector, hence the problem has the following boundary conditions:

$$b(\bar{r},t)\big|_{S'} = 0$$

$$b(\bar{r},t) \to 0 \qquad\qquad for \;\; |\bar{r}| \to \infty \qquad\qquad \textbf{(B. 2)}$$

The backwards Green's function $G(\bar{r},t|\bar{r}',t')$ for the scalar wave equation in half space medium without losses is given by

$$G = g(x,y,z,t|x',y',z',t') - g(x,y,z,t|x',y',-z',t') \qquad\qquad \textbf{(B. 3)}$$

with

$$g(\bar{r},t|\bar{r}',t') = \frac{d\,(t - t' + |\bar{r} - \bar{r}'|/v)}{|\bar{r} - \bar{r}'|} \tag{B. 4}$$

the Green's function for an imploding point source in a homogeneous medium. $v = 1/\sqrt{me}$ is the propagation velocity of the medium.

The field solution $b(\bar{r},t)$ is given by the integral Kirchhoff [1], derived from Green's theorem as

$$b(\bar{r},t) = -\frac{1}{4p}\int dt' \oiint_{S'} b(\bar{r}',t')\frac{\partial G}{\partial n'} - G\frac{\partial}{\partial n'}b(\bar{r}',t')dS' \tag{B. 5}$$

where n' is the vector normal to the surface S' and pointing outwards. The term $\frac{\partial G}{\partial n'}$ in (B.5) can be written as

$$\frac{\partial G}{\partial n'} = 2\frac{\partial g}{\partial n'} \tag{B. 6}$$

Introducing (B.6) together with (B.3) into (B5), the latter equation can be simplified to

$$b(\bar{r},t) = -\frac{2}{4p}\int dt' \oiint_{S'} b(\bar{r}',t')\frac{\partial g}{\partial n'}dS' \tag{B. 7}$$

The partial derivative of the Green's function $g(\bar{r},t|\bar{r}',t')$ equals

$$\frac{\partial g}{\partial n'} = \frac{\partial g}{\partial z'} = \frac{1}{|\bar{r}-\bar{r}'|}\frac{\partial d\,(t-t'+|\bar{r}-\bar{r}'|/v)}{\partial z'} + O(\frac{1}{|\bar{r}-\bar{r}'|^2})$$

$$= \frac{-1}{v|\bar{r}-\bar{r}'|}\frac{\partial d\,(t-t'+|\bar{r}-\bar{r}'|/v)}{\partial t'}\frac{d|\bar{r}-\bar{r}'|}{dz'} + O(\frac{1}{|\bar{r}-\bar{r}'|^2}) \qquad \textbf{(B. 8)}$$

If necessary, the second order term in $|\bar{r}-\bar{r}'|^{-2}$ can be taken into account [2], but in the far field the second order term is neglected.

The term $\frac{d|\bar{r}-\bar{r}'|}{dz'}$ can be transformed in

$$\frac{d|\bar{r}-\bar{r}'|}{dz'} = \frac{(z-z')}{|\bar{r}-\bar{r}'|} = \cos(q) \qquad \textbf{(B. 9)}$$

with the angle q as defined in Fig. 1.

Substituting (B.8) and (B.9) in equation (B.7), the field solution $b(\bar{r},t)$ becomes

$$b(\bar{r},t) = \frac{2}{4p\,v}\int dt' \oiint_{S'} b(\bar{r}',t')\frac{\partial d\,(t-t'+|\bar{r}-\bar{r}'|/v)}{\partial t'}\frac{\cos(q)}{|\bar{r}-\bar{r}'|}\,dS'$$

$$= \frac{2}{4p\,v}\oiint_{S'}\frac{\partial b(\bar{r}',t+|\bar{r}-\bar{r}'|/v)}{\partial t'}\frac{\cos(q)}{|\bar{r}-\bar{r}'|}\,dS' \qquad \textbf{(B. 10)}$$

Finally, the migrated image is found from the wavefield $b(\bar{r},t)$ in (B.10) at time $t=0$, hence

$$\hat{O}(x,y,z) = \frac{2}{4p\,v}\oiint_{S'}\dot{b}(x',y',z'=0,|\bar{r}-\bar{r}'|/v)\frac{\cos(q)}{|\bar{r}-\bar{r}'|}\,dS' \qquad \textbf{(B. 11)}$$

B.2. Stolt migration

The method is, just like the Kirchhoff migration, based on the back-propagation of the scalar wave equation and can thereby best be explained using the exploding source model. Important here is that only upcoming waves are assumed. Note also that the propagation velocity used in the exploding source model is half the value of the true medium velocity.

Consider a scalar field component $b(x, y, z, t)$ resulting from an exploding source. This component has to satisfy the scalar wave equation

$$\nabla^2 b(x, y, z, t) - me \frac{\partial^2}{\partial t^2} b(x, y, z, t) = 0 \qquad \textbf{(B. 12)}$$

Applying a Fourier transformation on (B.12), with respect to the x, y and the t co-ordinate, results in

$$\frac{\partial^2}{\partial z^2} B(k_x, k_y, z, w) = (-\frac{w^2}{v^2} + k_x^2 + k_y^2) B(k_x, k_y, z, w) \qquad \textbf{(B. 13)}$$

The Fourier transformation along the x and y co-ordinates only makes sense if the propagation velocity does not vary in the x and y directions. In the Fourier transformation, the following sign convention is used. For the forward transformation, the sign of the argument in the exponential is negative if the variable is time and positive if the variable is space. So the 3 dimensional forward Fourier transformation of $b(x, y, z, t)$ is defined as

$$B(k_x, k_y, z, w) = \iiint b(x, y, z, t) e^{ik_x x + ik_y y - iwt} dxdydt \qquad \textbf{(B. 14)}$$

Defining a wavenumber k_z as

$$k_z = \mathrm{sgn}(w)\sqrt{\frac{w^2}{v^2} - k_x^2 - k_y^2} \qquad \textbf{(B. 15)}$$

and substituting (B.15) in (B.13), equation (B.13) has as general solution

$$B(k_x, k_y, z, w) = Ce^{ik_z z} + De^{-ik_z z} \qquad \textbf{(B. 16)}$$

The constants C and D in (B.16) are found using the following two boundary conditions:

- assuming only upward coming waves, the first constant becomes $C = 0$.
- for z=0, the Fourier transformation of the measured data is found, hence $D = B(k_x, k_y, 0, w)$.

Finally (B.16) becomes

$$B(k_x, k_y, z, w) = B(k_x, k_y, 0, w)e^{-ik_z z} \qquad \textbf{(B. 17)}$$

Equation (B.17) represents the Fourier transform of the wavefront at depth z. The migrated image will be the inverse Fourier transform of (B.17) at time $t = 0$:

$$\hat{O}(x, y, z) = b(x, y, z, 0) = \iiint B(k_x, k_y, 0, w)e^{-i(k_x x + k_y y + k_z z)} dk_x dk_y dw \qquad \textbf{(B. 18)}$$

Equation (B.18) is the general representation of the f-k migration, also called the *Phase Shift Migration.* The method can deal with variations of velocity in function of depth. In the special case where $v(z) = v = cst$, equation (B.18) can be further developed by a change of variables from dw to dk_z. According to the definition of k_z in (B.15)

$$w = -k_z v \sqrt{1 + \frac{k_x^2 + k_y^2}{k_z^2}} \qquad \textbf{(B. 19)}$$

hence dw can be written as:

$$dw = \frac{k_z v^2}{w} dk_z \qquad \textbf{(B. 20)}$$

Replacing dw in (B.18) by the expression in (B.20), the migrated image becomes

$$\hat{O}(x, y, z) = v^2 \iiint \frac{k_z}{w} B(k_x, k_y, 0, w) e^{-i(k_x x + k_y y + k_z z)} dk_x dk_y dk_z \qquad \textbf{(B. 21)}$$

The advantage of (B.21) over (B.18) is that (B.21) can be calculated using an inverse 3D Fourier transformation of the measured data $B(k_x, k_y, 0, w)$, scaled by $\frac{v^2 k_z}{w}$, *i.e.* the Jacobian of the transformation from w to k_z. This means a serious reduction of the number of floating point operations for the migration and a gain in calculation time.

Although the method seems to be simple, the implementation of the Stolts migration algorithm can be tricky. The Fourier transformation with respect to the x, y and the t co-ordinate of the measured data provides equally spaced samples of $B(k_x, k_y, 0, w)$ on a rectangular grid in the (k_x, k_y, w) domain. This data is mapped into the (k_x, k_y, k_z) domain by the change of variables given by (B.19), where k_z is a non-linear function of k_x, k_y and w, resulting in an unevenly spaced data set. This mapping is graphically represented in Fig. 2. For simplicity we consider only the 2D case with the co-ordinates x and t. Fig. 2 shows the mapping of the evenly spaced points in the (k_x, w) domain, identified by black squares, into the unevenly spaced points in the (k_x, k_z) domain, identified by circles. The unevenly spaced data set in the (k_x, k_z) domain represents a problem for the inverse Fourier transformation.

Therefore the unevenly spaced data set has to be interpolated to fit an evenly spaced rectangular grid. Interpolating from an unevenly spaced data set is complicated, but can in this case be easily avoided.

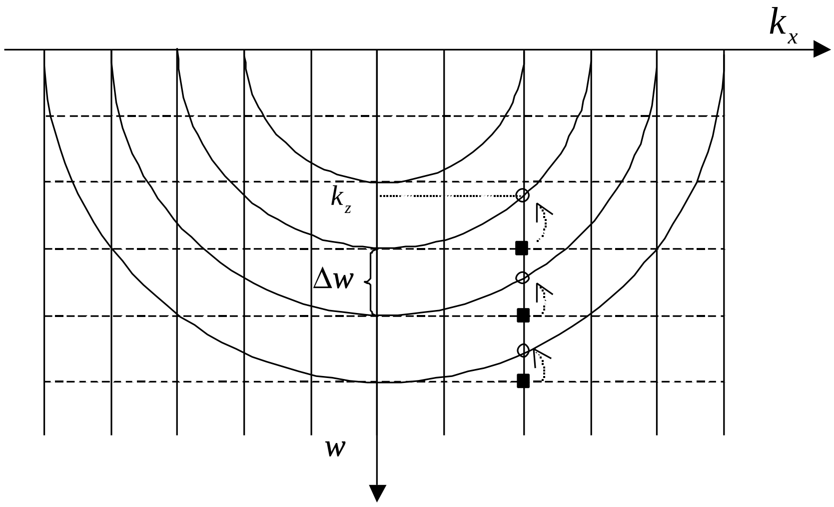

Fig. 2: Mapping in of the (k_x, w) domain into the (k_x, k_z) domain

Suppose the data in the (k_x, w) domain is spaced by Δw, and $w_n = n.\Delta w$. If the wavenumber vector k_z is put equal to the evenly spaced values w_n, the (k_x, k_z) domain has an evenly spaced grid. For each point (k_x, k_z) on the grid, the corresponding w can be calculated using (B.19). The value of w will probably be different from any of the values w_n. However $B(k_x, 0, w)$, needed for (B.21), can now be found by interpolating from an evenly spaced data $B(k_x, 0, w_n)$, which is less complicated than interpolating from unevenly spaced data. It is shown in [3] that the exact interpolation equation is

$$B(k_x, 0, w) = \sum_n B(k_x, 0, n\Delta w) h(w - n\Delta w) \qquad \textbf{(B. 22)}$$

where $h(w) = \mathrm{sinc}\,(\frac{w}{\Delta w})$ is the sinc interpolation function.

REFERENCES

[1] A. J. Berkhout, “Wave field extrapolation techniques in seismic migration, a tutorial,” *Geophysics*, vol. 46, no. 12, pp. 1638-1656, Dec. 1981.

[2] Ö. Yilmaz, *Seismic data processing*. Tulsa, USA: Society of Exploration Geophysicists, 1987, ch. 4.

[3] M. Soumekh, *Synthetic aperture radar signal processing*. New York: John Wiley & sons, 1999, ch. 4.